

# ON A QUESTION OF BUCHWEITZ ABOUT RANKS OF SYZYGIES OF MODULES OF FINITE LENGTH

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**ABSTRACT.** Let  $R$  be a local ring of dimension  $d$ . Buchweitz asks if the rank of the  $d$ -th syzygy of a module of finite length is greater than or equal to the rank of the  $d$ -th syzygy of the residue field, unless the module has finite projective dimension. Assuming that  $R$  is Gorenstein, we prove that if the question is affirmative, then  $R$  is a hypersurface. If moreover  $R$  has dimension two, then we show that the converse also holds true.

## 1. INTRODUCTION

Let  $(R, \mathfrak{m}, k)$  be a commutative Noetherian local ring with Krull dimension  $d$ . We consider the rank of the  $d$ -th syzygy of an  $R$ -module of finite length. We assume that  $R$  has positive depth, so that any  $R$ -module of finite length has a rank. On the ranks of syzygies, Buchweitz asks the following question [6, Question 11.16].

**Question 1.1** (Buchweitz). Does one have the equality

$$(1.1.1) \quad \text{rank}_R \Omega^d k = \min\{\text{rank}_R \Omega^d M \mid \text{pd}_R M = \infty \text{ and } \text{length}_R M < \infty\}?$$

Here we denote by  $\Omega^d M$  the  $d$ -th syzygy in the minimal free resolution of a finitely generated  $R$ -module  $M$ , and  $\text{pd}_R M$  stands for the projective dimension of  $M$ . If  $d = 1$ , then  $\Omega^d k$  has rank one, and Question 1.1 has an affirmative answer. Therefore, we consider the question for  $d \geq 2$ . Our main theorem is the following.

**Theorem 1.2.** *Assume  $R$  is Gorenstein and  $d \geq 2$ . Then Question 1.1 is affirmative only if  $R$  is a hypersurface.*

Here we say that  $R$  is a *hypersurface* if the  $\mathfrak{m}$ -adic completion of  $R$  is a quotient of a regular local ring by a regular element. This theorem says that if  $R$  is a Gorenstein local ring and not a hypersurface, then Question 1.1 has a negative answer.

On the other hand we can show the converse of Theorem 1.2 in the case  $d = 2$ .

**Theorem 1.3.** *Assume  $R$  is Gorenstein and  $d = 2$ . Then Question 1.1 is affirmative if and only if  $R$  is a hypersurface.*

This paper is organized as follows. In Section 2, we give a necessary condition for the equality (1.1.1) over a Gorenstein ring. In Section 3, we consider the Poincaré series of  $k$ , and prove Theorem 1.2 by using the necessary condition obtained in Section 2. Section 4 is devoted to proving Theorem 1.3 by using the notion of Buchsbaum-Rim complexes.

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## 2. A NECESSARY CONDITION FOR (1.1.1)

Throughout this section,  $(R, \mathfrak{m}, k)$  is a Gorenstein local ring of dimension  $d > 0$ . To prove Theorem 1.2, we use the following result which provides a necessary condition for the equality (1.1.1) to hold true.

**Proposition 2.1.** *There is an  $R$ -module  $M$  with  $\mathrm{pd}_R M = \infty$ ,  $\mathrm{length}_R M < \infty$ , and  $\mathrm{rank}_R \Omega^d M = \mathrm{rank}_R \Omega^{d-1} k + 1$ . Thus if Question 1.1 is affirmative, then there is an inequality*

$$(2.1.1) \quad \mathrm{rank}_R \Omega^d k \leq \mathrm{rank}_R \Omega^{d-1} k + 1.$$

In the rest of this section, we prove this proposition. First, we state the definition of a (minimal) MCM approximation.

**Definition 2.2.** (see [6, Chap. 11, Section 2]) For a finitely generated  $R$ -module  $M$ , an *MCM approximation* of  $M$  is a pair  $(X, p)$  of a maximal Cohen-Macaulay  $R$ -module  $X$  and a surjective homomorphism  $p : X \rightarrow M$  with  $\mathrm{pd}_R(\mathrm{Ker} p) < \infty$ . An MCM approximation  $(X, p)$  of  $M$  is called *minimal* if every  $\phi \in \mathrm{End}_R(X)$  with  $p \circ \phi = p$  is an automorphism.

Since  $R$  is Gorenstein, an (minimal) MCM approximation exists for any finitely generated  $R$ -module. We remark that an MCM approximation of  $M$  is unique up to free summands, and a minimal MCM approximation of  $M$  is unique up to isomorphism. We denote by  $X_M$  the maximal Cohen-Macaulay  $R$ -module in the minimal MCM approximation of  $M$ .

For an  $R$ -module  $M$  of finite length, we can construct  $X_M$  from the Matlis dual of  $M$  as follows (see the proof of [6, Proposition 11.15]).

**Lemma 2.3.** *Let  $M$  be an  $R$ -module of finite length. Then  $X_M \cong \mathrm{Hom}_R(\Omega^d \mathrm{Ext}_R^d(M, R), R)$ . In particular,  $\mathrm{rank}_R X_M = \mathrm{rank}_R \Omega^d \mathrm{Ext}_R^d(M, R)$ .*

The rank of the minimal MCM approximation of  $\Omega^{d-1} k$  is computed from that of  $\Omega^{d-1} k$ .

**Lemma 2.4.** *One has  $\mathrm{rank}_R X_{\Omega^{d-1} k} = \mathrm{rank}_R \Omega^{d-1} k + 1$ .*

*Proof.* Since  $M := \Omega^{d-1} k$  has depth  $d - 1$ , we have a short exact sequence

$$0 \rightarrow R^{\oplus r} \rightarrow X_M \rightarrow M \rightarrow 0$$

and  $r = \mathrm{length}_R \mathrm{Ext}_R^1(M, R) = \mathrm{length}_R \mathrm{Ext}_R^d(k, R) = 1$  by [6, Proposition 11.21]. Thus we have  $\mathrm{rank}_R X_M = \mathrm{rank}_R M + 1$ . ■

The rank of a maximal free summand of  $X_M$  is called the *(Auslander) delta invariant* of  $M$  and denoted by  $\delta_R(M)$ . We note that  $\delta_R(M)$  is well-defined without the Krull-Schmidt property of finitely generated  $R$ -modules. We give some properties of delta invariants in the next lemma.

**Lemma 2.5.** *Let  $M, N$  be finitely generated  $R$ -modules. The following hold.*

- (1) *If there exists a surjective homomorphism  $M \rightarrow N$ , then  $\delta_R(M) \geq \delta_R(N)$ .*
- (2) *If  $R$  is not regular, then  $\delta_R(\Omega^i k) = 0$  for all  $i \geq 0$ .*

*Proof.* See [6, Proposition 11.28] and [1, Proposition 5.7] respectively. ■

The following proposition plays a key role in the proof of Proposition 2.1.

**Proposition 2.6.** *There is an  $R$ -module  $M$  with  $\mathrm{pd}_R M = \infty$ ,  $\mathrm{length}_R M < \infty$ , and  $X_M \cong X_{\Omega_R^{d-1}k}$ .*

*Proof.* Let  $\underline{x} = x_1, \dots, x_d$  be a system of parameters of  $R$  with  $x_i \notin \mathfrak{m}^2 + (x_1, \dots, x_{i-1})$  for all  $i$ , and set  $R' = R/(\underline{x})$ . Put  $M$  to be the  $R$ -module  $\Omega_{R'}^{d-1}k$ . Then  $\mathrm{length}_R M < \infty$  and  $\mathrm{pd}_R M = \infty$  since  $\mathrm{length}_R R' < \infty$ ,  $\mathrm{pd}_R R' < \infty$ , and  $\mathrm{pd}_R k = \infty$ . We want to show that  $X_M \cong X_{\Omega_R^{d-1}k}$ . To prove this, it is enough to show that the following two claims hold.

**Claim 1.** Let  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  be an exact sequence of  $R$ -modules with  $\mathrm{pd}_R B < \infty$ . Then  $X_A \cong X_{\Omega C} \cong \Omega X_C$  up to free summands. Consequently,  $X_M \cong X_{\Omega_R^{d-1}k}$  up to free summands.

**Claim 2.** One has  $\delta_R(M) = 0 = \delta_R(\Omega_R^{d-1}k)$ .

*Proof of Claim 1.* There is an exact sequence  $0 \rightarrow \Omega B \rightarrow \Omega C \oplus P \rightarrow A \rightarrow 0$  with some free module  $P$ . Let  $W$  be a pull-back of  $\Omega B \rightarrow \Omega C \oplus P$  and  $p : X_{\Omega C} \rightarrow \Omega C$ . Then the induced sequences  $0 \rightarrow \mathrm{Ker} p \rightarrow W \rightarrow B \rightarrow 0$  and  $0 \rightarrow W \rightarrow X_{\Omega C} \rightarrow A \rightarrow 0$  are both exact. As  $\mathrm{pd}_R B < \infty$  and  $\mathrm{pd}_R(\mathrm{Ker} p) < \infty$ , we have  $\mathrm{pd}_R W < \infty$ . So  $X_{\Omega C}$  is an MCM approximation of  $A$  and isomorphic to  $X_A$  up to free summands. Applying this argument repeatedly, we get  $X_M \cong X_{\Omega_R(\Omega_{R'}^{d-2}k)} \cong \Omega_R X_{\Omega_{R'}^{d-2}k} \cong \Omega_R^2 X_{\Omega_{R'}^{d-3}k} \cong \dots \cong \Omega_R^{d-1} X_k \cong X_{\Omega_R^{d-1}k}$  up to free summands.  $\square$

*Proof of Claim 2.* By Lemma 2.5, it is enough to show that there is an epimorphism  $\Omega_R^{d-1}k \rightarrow M$ . We show this by induction on  $d$ . The case  $d = 1$  is trivial. So we assume  $d > 1$ . Put  $S = R/(x_1)$ . Since  $x_1 \in \mathfrak{m} \setminus \mathfrak{m}^2$ , one obtains  $\Omega_R^1 k \otimes_R S \cong k \oplus \Omega_S^1 k$ . So  $\Omega_R^{d-1}k \otimes_R S \cong \Omega_S^{d-2}k \oplus \Omega_S^{d-1}k$ . In particular, there is an epimorphism  $\Omega_R^{d-1}k \rightarrow \Omega_S^{d-1}k$ . By the hypothesis of induction, there is an epimorphism  $\Omega_S^{d-1}k \rightarrow M$ . So we have an epimorphism  $\Omega_R^{d-1}k \rightarrow M$ .  $\square$

The proof of the proposition is thus completed.  $\blacksquare$

Now we can give a proof of Proposition 2.1.

*Proof of Proposition 2.1.* By Proposition 2.6 and Lemma 2.4, there exists an  $R$ -module  $M$  with  $\mathrm{pd}_R M = \infty$ ,  $\mathrm{length}_R M < \infty$ , and  $\mathrm{rank}_R X_M = \mathrm{rank}_R \Omega^{d-1}k + 1$ . On the other hand,  $\mathrm{rank}_R X_M = \mathrm{rank}_R \Omega^d \mathrm{Ext}_R^d(M, R)$  by Lemma 2.6. Since  $M$  has finite length,  $M' = \mathrm{Ext}^d(M, R)$  also has finite length. Since  $\mathrm{pd}_R M = \infty$  and  $R$  is Gorenstein, we see that  $M'$  has infinite projective dimension. So  $M'$  satisfies the condition of Proposition 2.1, that is,  $\mathrm{pd}_R M' = \infty$ ,  $\mathrm{length}_R M' < \infty$ , and  $\mathrm{rank}_R \Omega^d M' (= \mathrm{rank}_R X_M) = \mathrm{rank}_R \Omega^{d-1}k + 1$ .  $\blacksquare$

### 3. THE POINCARÉ SERIES OF THE RESIDUE FIELD

Throughout this section,  $(R, \mathfrak{m}, k)$  is a local ring with  $\mathrm{depth} R > 0$ . So any  $R$ -module of finite length has rank 0. For a finitely generated  $R$ -module  $M$ , we denote by  $\beta_i(M)$  the  $i$ -th Betti number of  $M$ . Then the formal power series  $P_M(t) := \sum_{i=0}^{\infty} \beta_i(M)t^i$  is called the *Poincaré series* of  $M$ . Since  $\beta_0(k) = 1$ , there are integers  $\varepsilon_i$  and an equality

$$P_k(t) = \frac{\prod_{i=1}^{\infty} (1 + t^{2i-1})^{\varepsilon_{2i-1}}}{\prod_{j=1}^{\infty} (1 - t^{2j})^{\varepsilon_{2j}}};$$

see [2, Remark 7.1.1]. For example, it holds that  $\varepsilon_1 = \beta_1(k) = \mathbf{edim} R$  and  $\varepsilon_2 = \beta_2(k) - \binom{\beta_1(k)}{2}$ , where  $\mathbf{edim} R$  stands for the embedding dimension of  $R$ . By [2, Corollary 7.1.4], we have  $\varepsilon_i \geq 0$  for all  $i$ . So there is a formal power series  $Q(t) \in \mathbb{Z}[[t]]$  with non-negative coefficients and  $Q(0) = 1$  such that

$$P_k(t) = \frac{(1+t)^{\varepsilon_1}}{(1-t^2)^{\varepsilon_2}} Q(t).$$

The equality  $\mathbf{rank}_R \Omega^i k + \mathbf{rank}_R \Omega^{i+1} k = \beta_i(k)$  yields

$$\sum_{i=1}^{\infty} (\mathbf{rank}_R \Omega^i k) t^i = \frac{t}{1+t} P_k(t).$$

So we have

$$\begin{aligned} (3.0.1) \quad \sum_{i=1}^{\infty} (\mathbf{rank}_R \Omega^i k - \mathbf{rank}_R \Omega^{i-1} k) t^i &= (1-t) \sum_{i=1}^{\infty} (\mathbf{rank}_R \Omega^i k) t^i \\ &= t \frac{1-t}{1+t} P_k(t) = t \frac{(1+t)^{\varepsilon_1-2}}{(1-t^2)^{\varepsilon_2-1}} Q(t). \end{aligned}$$

From this equation, the main proposition of this section is deduced.

**Proposition 3.1.** *The inequality*

$$\mathbf{rank}_R \Omega^d k \leq \mathbf{rank}_R \Omega^{d-1} k + 1$$

*implies that  $R$  is a hypersurface or that  $d = 1$ .*

*Proof.* Since completion does not change the Betti numbers of  $k$ , we may assume that  $R$  is complete. Then  $R$  admits a presentation  $R = S/I$  with a regular local ring  $(S, \mathfrak{n})$  and an ideal  $I \subset \mathfrak{n}^2$  of  $S$ . By [3, Theorem 2.3.2], the number  $\varepsilon_2 = \beta_2(k) - \binom{\mathbf{edim} R}{2}$  is equal to  $\beta_0^S(I)$ . Now we assume that  $R$  is not a hypersurface and  $d \geq 2$ . Therefore one has  $\varepsilon_1 = \mathbf{edim} R \geq d + 2$  and  $\varepsilon_2 \geq 2$ . The formal power series  $Q'(t) = \frac{1}{(1-t^2)^{\varepsilon_2-1}} Q(t)$  also has non-negative coefficients and satisfies  $Q'(0) = 1$ , because  $\varepsilon_2 \geq 1$ . As a consequence of these observations, we see that

$$\begin{aligned} \text{the } d\text{-th coefficient of } t \frac{(1+t)^{\varepsilon_1-2}}{(1-t^2)^{\varepsilon_2-1}} Q(t) &= \text{the } (d-1)\text{-th coefficient of } (1+t)^{\varepsilon_1-2} Q'(t) \\ &\geq \binom{\varepsilon_1-2}{d-1} \geq \binom{d}{d-1} \geq d \geq 2. \end{aligned}$$

Combining this with the equation (3.0.1), we obtain  $\mathbf{rank}_R \Omega^d k - \mathbf{rank}_R \Omega^{d-1} k \geq 2$ . ■

Now we can easily see that Proposition 2.1 and 3.1 implies Theorem 1.2.

*Proof of Theorem 1.2.* Assume that Question 1.1 has an affirmative answer. Proposition 2.1 yields that the inequality  $\mathbf{rank}_R \Omega^d k \leq \mathbf{rank}_R \Omega^{d-1} k + 1$  holds. Then the ring  $R$  needs to be a hypersurface because of the consequence of Proposition 3.1. ■

## 4. THE CASE OF DIMENSION TWO

The aim of this section is to prove Theorem 1.3. In this section,  $(R, \mathfrak{m}, k)$  is a Cohen-Macaulay local ring of dimension  $d > 0$ . Let  $M$  be an  $R$ -module of finite length and  $\phi : R^{\oplus n} \rightarrow R^{\oplus m}$  be a homomorphism of free  $R$ -modules such that  $\text{Coker } \phi = M$ . We denote by  $I_m(\phi)$  the ideal of  $R$  generated by  $m$ -minors of  $\phi$ . Taking a non-maximal prime ideal  $\mathfrak{p}$  of  $R$ , we have  $M_{\mathfrak{p}} = 0$ . So  $\phi_{\mathfrak{p}} : R_{\mathfrak{p}}^{\oplus n} \rightarrow R_{\mathfrak{p}}^{\oplus m}$  is surjective and  $n \geq m$ . Moreover,  $(I_m(\phi))_{\mathfrak{p}} = I_m(\phi_{\mathfrak{p}})$  is equal to  $R_{\mathfrak{p}}$ . Consequently,  $I_m(\phi)$  is an  $\mathfrak{m}$ -primary ideal of  $R$ .

To prove Theorem 1.3, we want to estimate the rank of  $\Omega^2 M$ . It follows immediately from the exactness of  $0 \rightarrow \Omega^2 M \rightarrow R^{\oplus n} \rightarrow R^{\oplus m} \rightarrow M \rightarrow 0$  that  $\text{rank}_R \Omega^2 M = n - m$ . We can evaluate the number  $n - m$  from the next two propositions.

**Proposition 4.1.** [5, Theorem 3]. *Let  $\phi : R^{\oplus n} \rightarrow R^{\oplus m}$  be a homomorphism of free  $R$ -modules. Then for each integer  $0 \leq t \leq \min\{m, n\}$ , we have  $\text{ht } I_t(\phi) \leq (m-t+1)(n-t+1)$ .*

**Proposition 4.2.** [4, Corollary 2.7]. *Let  $\phi : R^{\oplus n} \rightarrow R^{\oplus m}$  be a homomorphism of free  $R$ -modules. Assume  $n \geq m$ . If  $\text{grade } I_m(\phi) = m - n + 1$ , then  $\text{pd}_R(\text{Coker } \phi) = m - n + 1$ .*

Using these proposition, we can give a proof of Theorem 1.3.

*Proof of Theorem 1.3.* We recall that the ideal  $I_m(\phi)$  is  $\mathfrak{m}$ -primary. Proposition 4.1 yields that  $n - m + 1 \geq \text{ht } I_m(\phi) \geq \text{grade } I_m(\phi) = d$ . Proposition 4.2 says that if  $M$  has infinite projective dimension, then  $d \neq n - m + 1$ . So we have  $d < n - m + 1$ . This inequality is equivalent to the inequality  $d \leq n - m (= \text{rank}_R \Omega^2 M)$ . The argument above implies that

$$d \leq \min\{\text{rank}_R \Omega^2 M \mid \text{pd}_R M = \infty \text{ and } \text{length}_R M < \infty\}.$$

Now we assume that  $R$  is a hypersurface and  $d = 2$ . Then  $\text{rank } \Omega^2 k = \beta_1(k) - \beta_0(k) = \text{edim } R - 1 = 2$ . The inequality above shows that Question 1.1 is affirmative.

The “only if” part follows from Theorem 1.2. ■

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